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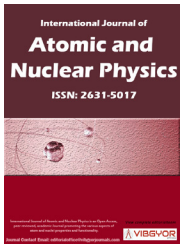
Symplectic Symmetry and Clustering in Atomic Nuclei



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Abstract

A new symplectic-based shell-model approach to clustering in atomic nuclei is proposed by considering the simple system ^{20}Ne. Its relation to the collective excitations of this system is mentioned as well. The construction of the Pauli allowed Hilbert space of the cluster states with maximal permutational symmetry is given for the ^{16}O + ^{4}He → ^{20}Ne channel in the case of one-component many-particle nuclear system. The equivalence of the obtained cluster model space to that of the semi-microscopic algebraic cluster model is demonstrated.

Keywords

Symplectic symmetry, Clustering in nuclei, Algebraic symplectec-based shell-model approach

Collective and Intrinsic Excitations

Symplectic structures arise in the Hamiltonian formulation of classical mechanics and its consecutive geometric quantization [1]. Symplectic geometry has revealed some symmetries of the Hamiltonian system, such as the symplectic structure and so on [2], as well as it has introduced the structure-preserving requirements into the numerical analysis [3] of dynamic systems. The structure-preserving method has been generalized to finite-dimensional, both conservative [4] and non-conservative [5] systems. This approach has been widely applied in the non-conservative dynamic systems and the good numerical behaviors have been verified in detail [6-9].

In the present work, however, we start with the quantum-mechanical treatment of the many-body nuclear system. Consider a one-component nuclear system consisting of *A* nucleons. Thus the nuclear many-body problem has *3A* degrees of freedom. Let us denote the position and momentum observables of all particles by x_{is} and $p_{is} = -i\hbar\partial / \partial x_{is}$ $(i = 1,2,3; s = 1,\ldots,A)$ [10], satisfying the standard commutation relations

$$[x_{is},\ p_{jt}] = i\hbar\delta_{ij}\delta_{st}, \qquad (1)$$

Then all Hermitian bilinear combinations of the position and momentum operators

$$x_{is}\,x_{jt}\ ,x_{is}\,p_{jt} + p_{jt}\,x_{is}\ ,\qquad p_{is}\,p_{jt} \qquad (2)$$

close under commutation and represent the generators of the symplectic group Sp(6A,R) - dynamical group of the problem of *3A* degrees of freedom. In Ref. [10] it has been shown that Sp(2N,R) is the group of linear canonical transformations in *2N*-dimensional phase space, being also a dynamical group of the *N*-dimensional

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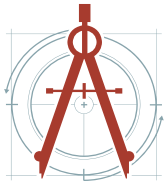

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harmonic oscillator. (For the case of *A* particles in 3-dimensional oscillator, *N* = *3A*.). The vector (x_{is} , p_{is}) of the many-particle nuclear phase space R^{6A} can be considered as a point of the manifold with a Lie algebra structure provided by the basic commutator (1) of the Heisenberg-Weyl algebra HW(3A). The Hilbert space of the nucleus will then be a carrier space for the unitary representations of the symplectic dynamical group. The symplectic symmetry therefore naturally arises from the consideration of the symplectic geometry of the many-particle nuclear system. In this way the unitary irreducible representations of the symplectic groups will play a fundamental role in the understanding of the quantum-mechanical properties of atomic nucleus. Often, to avoid the problem of center-of-mass motion from the very beginning, it is convenient to restrict the range of indices *s*, *t* from 1 to *A*-1 in Eqs.(1)-(2) and, hence, to refer x_{is} and p_{is} as to the relative Jacobi position and momentum coordinates of *m* = *A*-1 quasiparticles.

May be the first authors who have pointed out that the collective excitations are related to the symplectic groups as phenomenological dynamical groups are Goshen and Lipkin, which have considered in detail the one- [11] and two-dimensional [12] cases. After them, on the phenomenological level the group Sp(6,R) has been proposed as a dynamical group of collective excitations in nuclear system by P. Raychev [13,14] and later considered in detail on the microscopic level by G. Rosensteel and D. Rowe [15,16].

From the other side, it was shown that the collective effects are associated with operators that are scalar in O(m) and the collective Hamiltonian is obtained by projecting the many-particle Hamiltonian on a definite O(m) irrep associated with the *m* Jacobi vectors in the configuration space ^{3m} [17-21]. Asherova, et al. [22] have shown within the framework of the generalized hyperspherical functions method (GHFM) that in its minimal approximation, in which one restricts itself to a single O(m) irreducible representation (ω_1, ω_2, ω_3) of states of hyperspherical function within the lowest harmonic oscillator shell, is equivalent to restricting to a single SU(3) representation (ω_1-ω_2, ω_2-ω_3). They also showed, by means of an important $Sp(6,R)\otimes O(m)$ complementarity theorem of Moshinsky and Quesne [10], that the diagonalization of the GHFM Hamiltonian in the minimal approximation is equivalent to its diagonalization in the collective space of a single Sp(6,R) irrep. In this way the equivalence of the O(m)-based and symplectic-based theories of nuclear collective motion was quickly realized by many authors, e.g. [19-21,23,24]. Further, it was shown in detail, e.g., by Filippov and collobarators [19] that the group O(m) in the $Sp(6,R)\otimes O(m)$ is related to the intrinsic motion of the relative Jacobi quasiparticles with respect to the intrinsic principal axes frame of the mass quadrupole tensor, on which the collective excitations are built up. The group O(m) is also of importance because, as we will see further, it allows one to ensure the proper permutational symmetry of the nuclear wave functions.

The above considerations can be reformulated more generally. We can say that the set of basis states of the full dynamical symmetry group Sp(6m,R) of the whole many-particle nuclear system contains different kinds of possible motions - collective, intrinsic, cluster, etc. However, often, one restricts himself to a certain type of dominating excitation modes in the process under consideration. Thus, by reducing the group Sp(6m,R) one performs the separation of the *3m* nuclear variables {q} into kinematical (internal) and dynamical (collective) ones, i.e. {q} = {q_D, q_K}. The choice of the reduction chain depends on the concrete physical problem we want to consider. As we have mentioned, e.g., the group Sp(6,R) plays an important role in the treatment of the collective excitations in the one-component many-particle nuclear system. The reduction $Sp(6m,R)\supset Sp(6,R)\otimes O(m)$, which is obtained from Eq.(2) by contraction with respect to the indices *i* and *s*, thus turns out to be of a crucial importance in the microscopic nuclear theory of collective motions. In this way the considered reduction corresponds to the splitting of the microscopic many-particle configuration space R^{3m}, spanned by the relative Jacobi vectors, into kinematical and dynamical submanifolds, respectively. According to this, the many-particle nuclear wave functions can be represented respectively as consisting of collective and intrinsic components.

$$\Psi = \sum_\eta \Theta_\eta(q_D)\chi_\eta(q_K), \qquad (3)$$

Where $\chi_\eta|0\rangle = N_0(\lambda_0,\mu_0);KLM\rangle$ determines the bandhead structure, and the collective function Θ_η, in second quantized form, can be written as a polynomial in the Sp(6,R) raising operators.

Table 1: The Sp(6,R) representation $\langle\sigma\rangle = \langle 12+19/2,\ 4+19/2,\ 4+19/2\rangle$ of ^{20}Ne.

...
(12,0), (10,1), 2(8,2), (6,3), (7,1), (4,4), (6,0)
(10,0), (8,1), (6,2)
(8,0)

As will see further, by considering another reduction chain of the Sp(6m,R) group one is able to isolate the cluster degrees of freedom within the unified framework of the symplectic-based shell-model approach to nuclear excitations. To be more specific, we restrict ourselves to the case of two-cluster system only and consider the simplest case of ^{20}Ne system, which consists of two structureless (closed-shell) ^{16}O and α clusters. Thus, for example, the intrinsic wave function of maximal space symmetry which determines the intrinsic motion in ^{20}Ne has the O(m) symmetry (12,4,4). The collective excitations are then generated by acting with the Sp(6,R) raising generators on the complementary Sp(6,R) symplectic bandhead with SU(3) symmetry (8,0). The SU(3) basis states of the so obtained Sp(6,R) irreducible representation for ^{20}Ne are given in Table 1.

The ^{16}O + ^{4}HE → ^{20}NE Channel

Using the simple ^{16}O + α cluster system ^{20}Ne, in particular, we will prove the equivalence of the semi-microscopic algebraic cluster model (SACM) [25,26] and the one-component symplectic-based scheme in the classification of the cluster states in the many-particle nuclear Hilbert space.

The semi-microscopic algebraic cluster model

The semi-microscopic algebraic cluster model [25,26] was proposed as an approach to the cluster structure of light nuclei. In the SACM, the relative motion of the clusters is described by the vibron model [27], whereas their internal structure is treated in terms of the Elliott shell model, having a symmetry group $U_{ST}(4)\otimes U_C(3)$, where $U_C(3)$ is the symmetry group of the three-dimensional harmonic oscillator [28,29] and $U_{ST}(4)$ is the spin-isospin group [30]. The model space is constructed in a microscopic way by respecting the Pauli principle, but the interactions are expressed in terms of algebra generators. The states within the SACM for two-cluster system are then classified by the following reduction chain [25,26]:

$$U_{ST,1}(4)\otimes U_{C1}(3)\otimes U_{ST,2}(4)\otimes U_{C2}(3)\otimes U_R(4)$$

$$\supset U_{C1}(3)\otimes U_{C2}(3)\otimes U_R(3)\supset U_C(3)\otimes U_R(3) \qquad (4)$$

$$\supset U(3)\supset SU(3)\supset SO(3).$$

The spin-isospin irreducible representations for ^{16}O + ^{4}He → ^{20}Ne channel are given by $\tilde{f}$ = [4444], [1111] [5555], respectively. The SU(3) intrinsic structure for ^{16}O and α clusters, and combined system is determined by the scalar, or equivalent to it, irreps of the groups: $U_{C1}(3)$: [4, 4, 4] ~ [0, 0, 0], $U_{C2}(3)$: [0, 0, 0], and $U_C(3)$: [4, 4, 4] ~ [0, 0, 0]. Then a complete set of quantum numbers that characterize the cluster model states is defined by the following chain:

$$\begin{array}{ccccccc} U_{C1}(3)\otimes U_{C2}(3)\otimes U_R(3) & \supset & U_C(3)\otimes U_R(3) & & & & \\ [4,4,4]\ \ [0,0,0] & & [n,0,0] & & [4,4,4] & & [n,0,0] \\ \supset U(3) & \supset & SU(3) & \supset & SO(3), & & \\ [4+n,4,4] & & (n,0) & & L & & \end{array} \qquad (5)$$

where for closed-shell nuclei the $SU_{ST,i}(4)$ groups were dropped. Thus, the cluster model space is spanned by the SU(3) irreps (n_0 + n, 0), where *n* = 0, 1, 2, … . The Pauli principle requires n_0 = 8.

The one-component symplectic-based approach

At this point we want to point out that the cluster excitations, together with the collective and intrinsic motions, are naturally contained in the full dynamical group Sp(6m,R) of the many-nucleon nuclear system. In the case of two-cluster system ($A = A_1 + A_2$), the well-known anzatz [31].

$$\Psi = A\{\phi_1(A_1\text{-}1)\phi_2(A_2\text{-}1)f(q_0)\} \qquad (6)$$

can be related to the symplectic scheme by considering the following reduction chain:

$$Sp(6(A\text{-}1),R)$$

$$\supset Sp(6A_1,R)\otimes Sp(6(A_2\text{-}1),R) \qquad (7)$$

$$\supset Sp(6,R)_0\otimes Sp(6(A_1\text{-}1),R)\otimes Sp(6(A_2\text{-}1),R),$$

where the groups Sp(6(A_1-1),R) and Sp(6(A_2-1),R) describe the intrinsic state of the first and second cluster, respectively. One of the (A-1) Jacobi vectors, denote it by q_0 (from the set A_1), will

describe the relative motion of the two clusters, whereas the rest (A-2) Jacobi vectors will be related to the intrinsic states of the clusters. Thus, the group $Sp(6,R)_0$ in (7) will describe the "cluster excitations", related to the relative distance vector q_0. The group $Sp(6(A_1\text{-}1),R)$, describing the first cluster states, can be further reduced to $Sp(6,R)_{A1\text{-}1}\otimes O(A_1\text{-}1)$, i.e. $Sp(6(A_1\text{-}1),R)\supset Sp(6,R)_{A1\text{-}1}\otimes O(A_1\text{-}1)$, where as usual the first group in the direct product describes the collective excitations, whereas the second one - the intrinsic state of the cluster, on which the collective excitations are built up. Due to the mutually complementarity relationship [10], the $Sp(6,R)_{A1\text{-}1}$ irrep σ is completely determined by the irrep ω of $O(A_1\text{-}1)$ and vice versa. Similar considerations are valid for the second cluster with $(A_2\text{-}1)$ Jacobi quasiparticles. The wave function (6) in this case can be easily obtained from Eq.(3) by identifying $\chi_\eta(q_K)=\phi_1(A_1\text{-}1)\phi_2(A_2\text{-}1)$ and $\Theta_\eta(q_D)=f(q_0)$.

To relate the present classification scheme to that of SACM we consider further the reduction of the subgroups in Eq.(7), i.e. the complete reduction chain:

$$
\begin{aligned}
&Sp(6(A\text{-}1),R)\\
&\supset Sp(6,R)_0\otimes Sp(6(A_1\text{-}1),R)\otimes Sp(6(A_2\text{-}1),R)\\
&\supset Sp(6,R)_0\otimes Sp(6,R)_{A1\text{-}1}\otimes O(A_1\text{-}1)\\
&\qquad\otimes Sp(6,R)_{A2\text{-}1}\otimes O(A_2\text{-}1)\\
&\supset U(3)_0\otimes U_{A1\text{-}1}(3)\otimes U_{A2\text{-}1}(3)\otimes S_{A1}\otimes S_{A2}\\
&[n,0,0]\qquad [4,4,4]\qquad [0,0,0]\qquad f_1\qquad f_2\\
&\supset U(3)\supset SU(3)\supset SO(3).\\
&[4+n,4,4]\qquad (n,0)\qquad L
\end{aligned}
\tag{8}
$$

The permutational symmetry of the clusters is given by the reductions: O(15) ↓ S_{16}: (4, 4, 4) → {4, 4, 4, 4} and O(3) ↓ S_4: (0, 0, 0) → {4}. Then the permutational symmetry of the combined system will be f = {4, 4, 4, 4, 4}. Because of the full antisymmetry of the total wave function, the spin-isospin content of the combined system is given by conjugate Young scheme $\tilde{f}$ = [5, 5, 5, 5]. The cluster model space is then spanned by the even and odd $Sp(6,R)_0$ irreps respectively, i.e. by the sets of SU(3) irreps: (8,0), (10,0), (12,0), … and (9,0), (11,0), (13,0), …, which constitute a representation of the double covering metaplectic group Mp(6,R). Alternatively, the two $Sp(6,R)_0$ irreps could be unified into a single irrep of the direct-product group $[HW(3)_0]Sp(6,R)_0$. Note that because the $Sp(6,R)_0$ irreps are built up by a single Jacobi vector q_0, corresponding to the intercluster distance, only one-rowed irreps of the subgroup $SU(3)_0$ $Sp(6,R)_0$ are allowed of the type (n_0 +2n, 0). Thus, the positive-parity cluster state space in the ^{16}O + ^4He → ^{20}Ne channel within the one-component symplectic-based scheme will coincide with the Sp(6,R) irreducible collective space that is spanned by the fully symmetric SU(3) irreps only, given in Table 1 in red. Additionally, the full cluster model space with maximal permutational symmetry of SACM given by the SU(3) irreps (n_0 +n, 0) with n = 0, 1, 2, … is obtained by considering both the even and odd irreducible collective spaces of $Sp(6,R)_0$, in which the states of the three-dimensional harmonic oscillator of even and odd number of oscillator quanta fall. Thus, the cluster model spaces of the SACM and the one-component symplectic-based approach to the cluster states are identical. It is then clear that, based on the equivalence of the microscopic model spaces, the usage of the same (algebraic) Hamiltonian in both the SACM and one-component symplectic-based scheme to the clustering in atomic nuclei will produce identical spectra. We note also that the role of the Wigner supermultiplet group $SU_{ST,i}(4)$ (i=1,2), which is important in the construction of the Pauli allowed model space, is played in the symplectic-based scheme by the orthogonal group $O(A_i\text{-}1)$ through its reduction $O(A_i\text{-}1)\supset S_{Ai}$.

Conclusions

A new symplectic-based shell model approach to clustering in atomic nuclei is proposed. The cluster degrees of freedom are isolated by an appropriate separation of the full set of relative Jacobi many-particle variables into dynamical (collective) and kinematical (intrinsic) ones by reducing the full dynamical group Sp(6m,R) of the whole one-component (no distinction is made between the proton and neutron degrees of freedom) many-particle nuclear system. According to this, the nuclear wave functions are represented as having collective and intrinsic components. The kinematical part allows to ensure all the integrals

of motion of the considered nuclear system, including the proper permutational symmetry. The symplectic symmetry thus provides the nuclear cluster systems with fully microscopic shell-model wave functions that respect the Pauli principle.

For simplicity, the proposed algebraic approach is illustrated for the case of two-cluster nuclear systems by considering the simple system ^{20}Ne. The construction of Pauli allowed Hilbert space of the cluster states with maximal permutational symmetry is worked out for the $^{16}O + {}^{4}He \rightarrow {}^{20}Ne$ channel in the case of one-component many-particle nuclear system. The equivalence of the obtained cluster model space to that of the semi-microscopic algebraic cluster model is demonstrated. Further all the symplectic-based computational machinery can be used in practical applications. In contrast to the semi-microscopic algebraic cluster model, the symplectic Sp(6,R) symmetry allows to build up the required quadrupole collectivity observed in some nuclei without the use of an effective charge. Thus, the present approach can be tested in obtaining the excitation levels, including the microscopic structure of the cluster states and the transition probabilities, not only in ^{20}Ne, but also in other light nuclei for which the clustering is supposed to play an important role.

The relation of the present symplectic-based shell model approach to the collective excitations in ^{20}Ne is mentioned as well. We note also that the cluster motion is a relative motion of the one group of nucleons in phase with respect to the another group, i.e. it is also collective. In this regard, the cluster motion appears as a specific kind of collective excitations. Indeed, as we have demonstrated in the present work, the cluster model state space is a restricted part of the corresponding irreducible collective space of the two-cluster system as a whole.

Finally, the algebraic approach presented here could be generalized for more clusters or/and to the case of the two-component proton-neutron cluster nuclear systems.

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